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A SUBSTITUTE-KERNEL APPROXIMATION FOR RADIATIVE TRANSFER IN A NONGREY GAS NEAR EQUILIBRIUM, WITH APPLICATION TO RADIATIVE ACOUSTICS

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Abstract—The equations for radiative transfer in plane-parallel geometries are studied for a nongrey gas near equilibrium. Local thermodynamic equilibrium is assumed in the molecular processes. On the supposition that deviations from a reference state of radiative equilibrium are small, the equation of radiative transfer is linearized. This allows the required integrations over space and spectral frequency to be carried out independently. In analogy to the grey-gas procedures, a nongrey substitute-kernel (or exponential) approximation is then made for certain frequency-integrated transmission functions that occur in the expressions for the heat fluxes. This leads to a purely differential equation for the net radiative flux. The spectral properties appear in the formulation in two functions, which are introduced by the approximation and which depend on the reference state of the gas. These functions are found by analytical matching procedures, which define linearized Planck- and Rosseland-like mean absorption coefficients that are physically meaningful for a general nongrey gas. For use in radiative acoustics, the differential equation for the heat flux is coupled with the linearized equations of gas dynamics. The resulting nongrey equations have the same mathematical structure as the grey equations, which are now contained as a special case. The results of existing grey-gas solutions can therefore be reinterpreted in terms of a nongrey gas by an appropriate normalization.

NOMENCLATURE

a, b, constants in the grey substitute-kernel approximation;
a_{S_0}, isentropic speed of sound;
a_{T_0}, isothermal speed of sound;
B_ν, Planck function;
B_{T_0}, temperature derivative of the frequency-integrated Planck function [see equation (16)];
Bo, ordinary Boltzmann number [see equation (43)];
$\hat{B}o$, nongrey Boltzmann number [see equation (43)];
c_{p_0}, specific heat at constant pressure;
$E_n(z)$, exponential-integral function [see equation (8)];
$F_n(z)$, frequency-integrated transmission function [see equations (14) and (15)];
h, specific enthalpy;
I_ν, specific intensity;
l, direction cosine of direction of radiative propagation;
m_0, n_0, functions in the nongrey substitute-kernel approximation [see equation (20)];
p, pressure;
$Q^R_{\nu\pm}$, one-sided radiant heat fluxes per unit frequency;
$Q^R_\pm$, one-sided radiant heat fluxes;
q^R, net radiant heat flux, $Q^R_+ - Q^R_-$;
T, temperature;
t, time;

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u, velocity;
x, space coordinate;
z, general argument of a function;
α_ν, spectral absorption coefficient;
α_0, grey absorption coefficient;
α_{LP_0}, linear Planck mean emission coefficient;
α_R, Rosseland mean absorption coefficient;
ν, spectral frequency;
ξ, normalized space coordinate [see equation (35)];
ρ, density;
τ, normalized time [see equation (35)];
φ, potential function.

Subscripts
$(\ )_w$, quantity pertaining to the wall;
$(\ )_0$, evaluated at the reference condition.

Superscripts
$(\ )'$, perturbation quantity;
$(\hat{\ })$, quantity restricted to certain frequency ranges by the integration convention [see equation (10)];
$(\bar{\ })$, normalized quantity [see equation (35)].

1. INTRODUCTION

THE ASSUMPTION of a grey gas, though useful for exploratory purposes, does not lead to a sufficiently accurate description of radiative transfer for current experimental and theoretical studies of radiatively driven acoustic waves [1, 2]. The present theory has therefore been formulated to retain the essentially nongrey character of the transfer problem, while remaining simple enough for analytical solution of the resulting acoustic equations. The basic formulation is also applicable, however, to any plane-parallel transfer problem, static or dynamic, in which the temperature and density variations within the gas are small enough to allow linearization about an equilibrium reference state. The equations appropriate for such general application are therefore developed first without reference to gas dynamics. The acoustic theory that gave the original motivation is then discussed at the end.

The development assumes (see, for example, Vincenti and Kruger [3]) that nonequilibrium effects from all purely molecular processes are negligible. The radiative effects are thus taken into account on the hypothesis of local thermodynamic equilibrium.† Radiative scattering is neglected as being small in the applications in which we are ultimately interested. To fix the problem, the geometrical configuration is taken to be that of a semi-infinite expanse of radiating gas to the right of an infinite, plane, radiating black wall.

On the assumption that deviations from radiative equilibrium are small, the equation of radiative transfer is first linearized about an equilibrium reference state. This allows the necessary integrations over space and spectral frequency to be carried out independently. This fact was apparently first noted by Baldwin [5] but was not fully exploited by him. Certain of the present ideas in embryonic form have also appeared in the work by Ryhming [6], who considered acoustic propagation in a gas radiating in a grey band (absorption coefficient constant over a finite range of frequency). The following formulation, however, places no restrictions on the absorption coefficient.

Following the above integrations, the expression for the radiative heat flux appears in integral form. A nongrey substitute-kernel (or exponential) approximation is then made for certain frequency-integrated transmission functions, which account for both the spectral and directional properties of the radiative field. As in the analogous procedure for a grey gas, this approximation leads to a purely differential equation for the heat flux. This second-order

† The nongrey substitute-kernel approximation, which will be central to the formulation, is not limited to gases in local thermodynamic equilibrium and has been employed by Gilles [4] in a theory for coupled radiative and vibrational nonequilibrium. As an example of a nonacoustic application, it is used there to obtain a perturbation solution for steady flow through a normal shock wave.

equation contains the grey-gas equation as a special case and has, in fact, the identical mathematical structure as the grey equation. The same can be said regarding the radiative boundary condition, which is also derived. By introduction of an appropriate normalization, the present results can therefore be used to reinterpret, for a nongrey gas, many of the previous grey-gas solutions.

The nongrey substitute-kernel approximation introduces two functions denoted by $m_0(\rho_0, T_0)$ and $n_0(\rho_0, T_0)$, where ρ_0 and T_0 are the undisturbed density and temperature, respectively. These functions can be related analytically or numerically to the spectral properties of the gas. They can be eliminated formally from the final equations through the definition of the normalized variables. To obtain results in terms of the physical variables, however, requires their specification. In the present work, this is done analytically by matching certain properties (i.e. zero intercept, area, or first moment) of the approximate exponential kernel to those of the corresponding exact transmission function. This results in the definition of linearized Planck- and Rosseland-like mean absorption coefficients that are physically meaningful for a general nongrey gas. For a gas in which the absorption coefficient is nonzero at all frequencies, these frequency-averaged coefficients reduce to the ordinary linearized Planck and Rosseland means discussed by Cogley, Vincenti and Gilles [7].

For application in radiative acoustics, the differential equation governing the radiative heat flux is finally coupled with the linearized equations of gas dynamics. This leads to a single fifth-order partial differential equation for a perturbation potential function. This equation and the corresponding radiative boundary condition also contain their grey-gas counterparts, with which they share a common mathematical structure.

2. EQUATIONS OF RADIATIVE TRANSFER NEAR EQUILIBRIUM

The coordinate system is shown in Fig. 1. The equation governing the frequency-dependent specific intensity I_ν can be written, with the relatively small time-derivative term omitted, as (see, for example, [3])

$$l\frac{\partial I_\nu}{\partial x} = \alpha_\nu[B_\nu - I_\nu]. \tag{1}$$

Here the subscript denotes values at the spectral frequency ν; α_ν is the volumetric absorption coefficient, B_ν the Planck function, $l \equiv \cos\phi$ the direction cosine of the direction of radiative

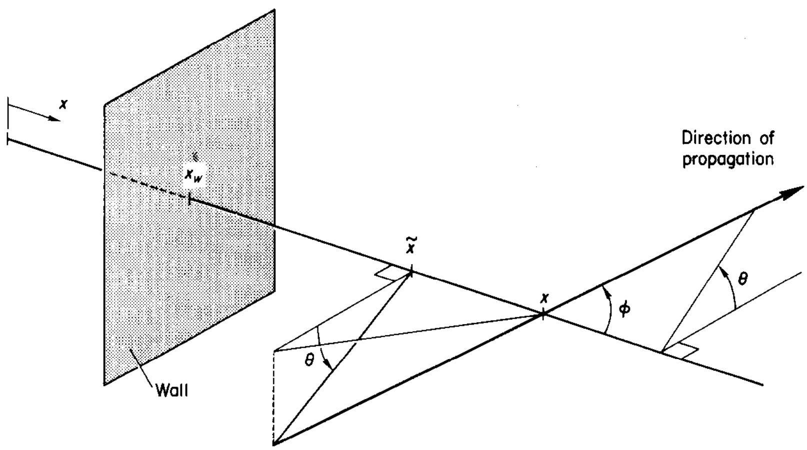

FIG. 1. Coordinate system.

propagation, and x the space coordinate normal to the wall. Equation (1) can be linearized about a reference state of radiative equilibrium to obtain the following equation for the perturbation specific intensity I'_ν:

$$l\frac{\partial I'_\nu}{\partial x} = \alpha_{\nu 0}[B'_\nu - I'_\nu]. \quad (2)$$

Subscript 0 denotes the equilibrium reference condition, and the primed quantities are small perturbations defined by $I_\nu \equiv I_{\nu_0} + I'_\nu = B_{\nu_0} + I'_\nu$ and $B_\nu \equiv B_{\nu_0} + B'_\nu = B_{\nu_0} + \mathrm{d}B_\nu/\mathrm{d}T|_0 T'$, where T is the temperature.

Implicit in our employment of a condition of radiative equilibrium for a semi-infinite expanse of gas is the notion that equations (2) through (7) are written only for those spectral frequencies for which $\alpha_{\nu_0} \neq 0$. Within such absorbing and emitting "bands," radiative equilibrium can exist, and the reference intensity is justifiably taken as B_{ν_0} irrespective of the direction l. For frequencies at which $\alpha_{\nu_0} = 0$, true equilibrium can not exist, since in the absence of a wall on the right there is no mechanism for emission at such frequencies in directions $l < 0$. B_{ν_0} is therefore not available as a reference in those directions. Such frequencies, however, are of no interest to us (even for $l > 0$), since they entail no coupling between the radiation and the gas. A frequency-integration convention will be introduced later that takes care of this matter in an automatic way.

The formal solution of equation (2) can be written (cf. [3], p. 481)

$$I'_\nu(x, l > 0) = I'_\nu(x_w, l) \exp[-\alpha_{\nu 0}(x - x_w)/l] + \int_{x_w}^{x} \alpha_{\nu 0} \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\bigg|_0 T' \exp[-\alpha_{\nu 0}(x - \tilde{x})/l]\,\mathrm{d}\tilde{x}/l \quad (3a)$$

and

$$I'_\nu(x, l < 0) = -\int_{x}^{\infty} \alpha_{\nu 0} \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\bigg|_0 T' \exp[-\alpha_{\nu 0}(x - \tilde{x})/l]\,\mathrm{d}\tilde{x}/l, \quad (3b)$$

where $\tilde{x}$ is the running variable of integration and the subscript w denotes quantities pertaining to the wall. The solution is written in two parts because of the different boundary conditions on I'_ν for different ranges of l. For propagation away from the wall ($l > 0$), we have $I'_\nu(x_w, l) = \mathrm{d}B_\nu/\mathrm{d}T|_0 T'_w$, since the wall is assumed to radiate as a black body. For propagation toward the wall ($l < 0$), the boundary condition is $I'_\nu(\infty, l) = 0$, since all perturbation quantities are taken to be zero at infinity.

The perturbations in the one-sided radiant heat fluxes per unit frequency for the positive and negative directions, respectively, are defined by (cf. [3], p. 441)

$$Q^{R'}_{\nu +} \equiv -2\pi \int_{+1}^{0} l\, I'_\nu(x, l > 0)\,\mathrm{d}l \quad (4)$$

and

$$Q^{R'}_{\nu -} \equiv 2\pi \int_{0}^{-1} l\, I'_\nu(x, l < 0)\,\mathrm{d}l. \quad (5)$$

Substituting solutions (3a, b) into equations (4) and (5) and carrying out mathematical details similar to those described in [3], p. 481, we obtain

$$Q^{R'}_{\nu +} = 2\pi\left\{\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\bigg|_0 T'_w E_3[\alpha_{\nu 0}(x - x_w)] + \int_{x_w}^{x} \alpha_{\nu 0}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\bigg|_0 T' E_2[\alpha_{\nu 0}(x - \tilde{x})]\,\mathrm{d}\tilde{x}\right\} \quad (6)$$

and

$$Q^{R'}_{\nu -} = 2\pi\left\{\int_{x}^{\infty} \alpha_{\nu 0}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\bigg|_0 T' E_2[\alpha_{\nu 0}(\tilde{x} - x)]\,\mathrm{d}\tilde{x}\right\}, \quad (7)$$

where the exponential-integral functions are defined by

$$E_n(z) \equiv \int_0^1 \exp(-z/l)\, l^{n-2}\,\mathrm{d}l, \quad (n \text{ an integer}). \quad (8)$$

These functions satisfy the recurrence formula

$$\frac{\mathrm{d}\,E_n(z)}{\mathrm{d}z} = -E_{n-1}(z). \quad (9)$$

We must now integrate equations (6) and (7) over frequency. To do this we introduce the integration convention

$$\hat{Q}^{R'}_{\pm} \equiv \int_{\alpha_{\nu 0} \neq 0} Q^{R'}_{\nu \pm} \, d\nu, \tag{10}$$

where the notation $\alpha_{\nu 0} \neq 0$ signifies an integration over all frequencies for which $\alpha_{\nu 0}$ is nonzero. In the special situation where $\alpha_{\nu 0}$ is nonzero for all ν, we have the correspondence

$$\int_{\alpha_{\nu 0} \neq 0} (\ ) \, d\nu \xrightarrow[\text{for all } \nu]{\alpha_{\nu 0} \text{ nonzero}} \int_0^\infty (\ ) \, d\nu. \tag{11}$$

The foregoing integration convention automatically selects that part of the perturbed emission from the black wall (the first term in equation (6)) that can interact with the gas. Wall emission at frequencies at which $\alpha_{\nu 0} = 0$ can never be absorbed by the gas and hence is of no consequence in determining the thermodynamic state of the gas.

The integration convention (10) is general and implies no restrictive assumptions concerning the absorption characteristics of the gas. The limits of the frequency integration are free to be set in each particular case by the nature of the absorption coefficient. The convention also allows all the usual mathematical manipulations of the resulting equations because their singular nature for $\alpha_{\nu 0} = 0$ has been eliminated. Still other reasons for introducing the convention are concerned with the substitute-kernel approximation and will be more easily understood in that context.

We now integrate equations (6) and (7) over frequency according to the convention (10). The frequency-integrated one-sided perturbation heat fluxes in the range $\alpha_{\nu 0} \neq 0$ are then

$$\hat{Q}^{R'}_{+} = 2\pi \left\{ T'_w \int_{\alpha_{\nu 0} \neq 0} \left. \frac{dB_\nu}{dT} \right|_0 E_3(\alpha_{\nu 0} x) \, d\nu + \int_0^x T' \left[\int_{\alpha_{\nu 0} \neq 0} \alpha_{\nu 0} \left. \frac{dB_\nu}{dT} \right|_0 E_2[\alpha_{\nu 0}(x - \tilde{x})] \, d\nu \right] d\tilde{x} \right\} \tag{12}$$

and

$$\hat{Q}^{R'}_{-} = 2\pi \left\{ \int_x^\infty T' \left[\int_{\alpha_{\nu 0} \neq 0} \alpha_{\nu 0} \left. \frac{dB_\nu}{dT} \right|_0 E_2[\alpha_{\nu 0}(\tilde{x} - x)] \, d\nu \right] d\tilde{x} \right\}, \tag{13}$$

where x_w has been taken as zero for convenience. The integration over frequency has been interchanged with that over $\tilde{x}$ in these equations to emphasize that the frequency integration can be carried out independently of the perturbations in the gas. This possibility is unique to the near-equilibrium situation, in which all frequency-dependent quantities are evaluated at the equilibrium reference state. When the gas is far from equilibrium, the corresponding nonlinear equations depend explicitly on B_ν and α_ν, which are functions of position as well as frequency. The integrations over the two variables can not then be carried out independently, as they can for the linearized equations.

We can exploit the independence of the frequency integration by defining the transmission functions

$$F_2(z; \rho_0, T_0) \equiv \frac{1}{\hat{B}_{T_0}} \int_{\alpha_{\nu 0} \neq 0} \alpha_{\nu 0} \left. \frac{dB_\nu}{dT} \right|_0 E_2(\alpha_{\nu 0} z) \, d\nu \tag{14}$$

and

$$F_3(z; \rho_0, T_0) \equiv \frac{1}{\hat{B}_{T_0}} \int_{\alpha_{\nu 0} \neq 0} \left. \frac{dB_\nu}{dT} \right|_0 E_3(\alpha_{\nu 0} z) \, d\nu. \tag{15}$$

The quantity $\hat{B}_{T_0}$ in these equations is defined by

$$\hat{B}_{T_0} \equiv \int_{\alpha_{\nu 0} \neq 0} \left. \frac{dB_\nu}{dT} \right|_0 d\nu \xrightarrow[\text{for all } \nu]{\alpha_{\nu 0} \text{ nonzero}} \frac{4\sigma T_0^3}{\pi} \equiv B_{T_0}, \tag{16}$$

where σ is the Stefan–Boltzmann constant. As a consequence of the recurrence formula (9), the functions F_n satisfy a similar relation, that is,

$$\frac{dF_n(z)}{dz} = -F_{n-1}(z). \tag{17}$$

Equations (12) and (13) can now be written in the more compact form

$$\hat{Q}_{+}^{R'} = 2\pi\,\hat{B}_{T_0}\{T'_w F_3(x) + \int_0^x T' F_2(x - \tilde{x})\,d\tilde{x}\}, \tag{18}$$

and

$$\hat{Q}_{-}^{R'} = 2\pi\,\hat{B}_{T_0}\{\int_x^\infty T' F_2(\tilde{x} - x)\,d\tilde{x}\}. \tag{19}$$

The definitions (14) and (15) of F_2 and F_3 cause equations (18) and (19) to look formally like their grey-gas counterparts. In particular, F_2, F_3, and $\hat{B}_{T_0}$ correspond respectively to $\alpha_0 E_2(\alpha_0 z)$, $E_3(\alpha_0 z)$, and $B_{T_0} = 4\sigma T_0^3/\pi$, where α_0 is the grey absorption coefficient. In fact, the former quantities reduce directly to the latter when $\alpha_{\nu_0} = \alpha_0 =$ constant for all ν in equations (14), (15), and (16). The nongrey formulation thus contains the grey formulation as a special case. Moreover, the mathematical structure of the nongrey equations is the same as for the grey equations. The implications of this will be discussed at the end of the next section.

The transmission functions F_2 and F_3 can be computed numerically, given data for $\alpha_{\nu_0}(\rho_0, T_0)$. They incorporate both the spectral and directional character of the radiative field into equations (18) and (19), irrespective of the variations in temperature. The expressions for the fluxes, however, are still in integral form. To obtain analytical solutions, particularly of the resulting acoustic equations, it is desirable to have the flux described by a purely differential equation. A substitute-kernel approximation for the nongrey gas will therefore be introduced.

3. NONGREY SUBSTITUTE-KERNEL APPROXIMATION

In grey-gas theory, a widely and successfully used approximation is to replace the exponential-integral kernel E_2 by a purely exponential function according to $E_2(\alpha_0 z) \cong a \exp(-b\alpha_0 z)$, where a and b are dimensionless constants. Noting that the transmission function F_2 has a strong dependence on E_2, Baldwin [5] suggested (although in a different formalism) that F_2 be replaced in analogous fashion. Following this suggestion, we therefore introduce the approximation

$$F_2(z; \rho_0, T_0) \cong m_0(\rho_0, T_0) \exp\{-n_0(\rho_0, T_0)\,z\}, \tag{20}$$

where the dependence on ρ_0 and T_0 is included in the functions m_0 and n_0, which thus have fixed values for any given reference condition. These functions represent, in effect, some as yet undefined, frequency-averaged emission and absorption coefficients, respectively. In order that the substitute kernel will satisfy the recurrence formula (17), we correspondingly replace F_3 by

$$F_3(z; \rho_0, T_0) \cong \frac{m_0}{n_0} e^{-n_0 z}. \tag{21}$$

We note in passing that it is the introduction of the integration convention (10) that makes it possible to replace F_3 by a pure exponential. If we had integrated from 0 to ∞, the counterpart of F_3 would have been defined and its exponential approximation written (since $E_3(0) = \frac{1}{2}$) as

$$\frac{1}{B_{T_0}} \int_0^\infty \left.\frac{dB_\nu}{dT}\right|_0 E_3(\alpha_{\nu_0} z)\,d\nu = \frac{1}{2B_{T_0}} \int_{\alpha_{\nu_0}=0} \left.\frac{dB_\nu}{dT}\right|_0 d\nu + \frac{1}{B_{T_0}} \int_{\alpha_{\nu_0}\neq 0} \left.\frac{dB_\nu}{dT}\right|_0 E_3(\alpha_{\nu_0} z)\,d\nu \cong f(T_0) + \frac{\hat{B}_{T_0}}{B_{T_0}} \frac{m_0}{n_0} e^{-n_0 z}.$$

The function $f(T_0)$ in effect accounts for that portion of the radiant heat flux from the black wall that cannot interact with the gas and thus does not enter into the present formulation. Definition (10) allows us to dispense with this unessential function.

The accuracy of the approximation (20) depends, of course, on how well a pure exponential, with suitable choice of m_0 and n_0, can represent the exact frequency-integrated transmission function. In principle, m_0 and n_0 could be determined by fitting the exponential (20) (e.g. by a least-squares method) to the results of a numerical calculation of F_2. Instead, we present in the next section an analytical matching procedure that will express m_0 and n_0 in terms of the aforementioned frequency-averaged emission and absorption coefficients. In either event, since m_0 and n_0 would most likely have to be evaluated specially for each different reference condition, the nongrey substitute-kernel approximation is different from the analogous approximation for a grey gas. In the latter case, E_2 is a function solely of the product $\alpha_0 z$; the dependence on the physical state of the gas is absorbed into this generalized argument through the value of α_0 (which must then be chosen on the basis of independent arguments). A general comparison of E_2 with its exponential approximation in terms of this generalized argument is therefore possible for a grey gas (see [3], p. 484, Fig. 2). Such comparison cannot be made for the nongrey substitute-kernel approximation.

The present formulation is different from that for a grey gas in another important aspect. The nongrey substitute-kernel approximation contains the two parameters m_0 and n_0 and the integration convention (10), which together allow us to characterize the absorption coefficient and radiative emission for a nongrey gas; that is, n_0 and $4\pi m_0 \hat{B}_{T_0} T'$ represent the absorption coefficient and the rate of spontaneous emission per unit volume, respectively (see, for example, equation (25) below). The grey substitute-kernel approximation, on the other hand, contains only the one parameter α_0, the grey absorption coefficient; the grey spontaneous emission follows necessarily as $4\pi a\alpha_0 B_{T_0} T' = 16 a\alpha_0 \sigma T_0^3 T'$. (The constants a and b in the grey substitute kernel must be of order unity and reflect the behavior of the function E_2, not the spectral properties of the gas.) We thus have in the present formulation more freedom with which to represent the radiative properties of the gas.

If, in the interest of greater accuracy, a sum of exponentials were assumed in place of the approximation (20) (e.g. $F_2(z; \rho_0, T_0) \cong \sum_j m_{0_j} \exp(-n_{0_j} z)$), an essentially many-parameter fit could be obtained for the transmission function.† One can, of course, also retain additional exponential terms in the grey-gas formulation, giving a many-parameter fit to the function E_2 (or more precisely $\alpha_0 E_2$). This does not, however, alter the fact that $\alpha_0 E_2(\alpha_0 z)$ is itself a poor approximation to begin with for the correct transmission function for a nongrey gas.

With the substitute-kernel approximation, equations (18) and (19) become

$$\hat{Q}_+^{R'} = 2\pi \hat{B}_{T_0} \left\{ T'_w \frac{m_0}{n_0} e^{-n_0 x} + \int_0^x T' m_0 \exp[-n_0(x - \tilde{x})] \, d\tilde{x} \right\} \tag{22}$$

and

$$\hat{Q}_-^{R'} = 2\pi \hat{B}_{T_0} \{ \int_x^\infty T' m_0 \exp[-n_0(\tilde{x} - x)] \, d\tilde{x} \}. \tag{23}$$

Instead of these one-sided heat fluxes, it will be more convenient to deal with the net heat flux $\hat{q}^{R'} \equiv \hat{Q}_+^{R'} - \hat{Q}_-^{R'}$. This is because we are interested primarily in the derivative $\partial \hat{q}^{R'}/\partial x$, which gives the net energy lost by the gas per unit volume. The equations for $\hat{q}^{R'}$ and $\partial \hat{q}^{R'}/\partial x$ follow from equations (22) and (23) as

$$\hat{q}^{R'} = 2\pi \hat{B}_{T_0} \left\{ T'_w \frac{m_0}{n_0} e^{-n_0 x} + \int_0^x T' m_0 \exp[-n_0(x - \tilde{x})] \, d\tilde{x} \right.$$

† This procedure has been carried through and leads to a higher-order partial differential equation than the one obtained here (two orders higher for each additional exponential retained).

$$- \int_x^\infty T' m_0 \exp[-n_0(\tilde{x} - x)] \, d\tilde{x} \Big\} \quad (24)$$

and

$$\frac{\partial \hat{q}^{R'}}{\partial x} = -2\pi \hat{B}_{T_0} \Big\{ T'_w m_0 e^{-n_0 x} + \int_0^\infty T' m_0 n_0 \exp[-n_0 |\tilde{x} - x|] \, d\tilde{x} - 2m_0 T' \Big\}. \quad (25)$$

As in the grey-gas case, the radiative-transfer problem can now be formulated in purely differential form. For this we need the second derivative of $\hat{q}^{R'}$, which is

$$\frac{\partial^2 \hat{q}^{R'}}{\partial x^2} = 2\pi \hat{B}_{T_0} \Big\{ T'_w m_0 n_0 e^{-n_0 x} + \int_0^x T' m_0 n_0^2 \exp[-n_0(x - \tilde{x})] \, d\tilde{x} - \int_x^\infty T' m_0 n_0^2 \exp[-n_0(\tilde{x} - x)] \, d\tilde{x} + 2m_0 \frac{\partial T'}{\partial x} \Big\}. \quad (26)$$

Subtracting n_0^2 times equation (24) from equation (26) then results in

$$\frac{\partial^2 \hat{q}^{R'}}{\partial x^2} - 4\pi m_0 B_{T_0} \frac{\partial T'}{\partial x} - n_0^2 \hat{q}^{R'} = 0, \quad (27)$$

which is a purely differential equation governing the net radiative heat flux $\hat{q}^{R'}$.

The appropriate radiative boundary condition at $x = 0$ can be found by evaluating equations (24) and (25) at $x = 0$. Doing this and eliminating the integral term between the resulting equations, we obtain

$$\left[\frac{\partial \hat{q}^{R'}}{\partial x} - n_0 \hat{q}^{R'} \right]_{x=0} = -4\pi m_0 \hat{B}_{T_0}(T'_w - T'_{x=0}). \quad (28)$$

Equation (28) relates the heat flux at the wall to the temperature jump $(T'_w - T'_{x=0})$. If we examine the physical meaning of each term in the equations leading to this relation, we find that it expresses the energy balance for an element of gas at $x = 0$. Our ability to write the energy balance as equation (28) is, of course, a consequence of the substitute-kernel approximation.

By working with the one-sided heat fluxes, we can obtain the radiative boundary condition from a different argument. We begin by evaluating the exact one-sided flux at the wall from equation (18). This gives

$$\hat{Q}_+^{R'}|_{x=0} = 2\pi \hat{B}_{T_0} T'_w F_3(0) = \pi \hat{B}_{T_0} T'_w. \quad (29)$$

This is then used in conjunction with the relation $\hat{q}^{R'} = \hat{Q}_+^{R'} - \hat{Q}_-^{R'}$ and equations (23) and (25), all evaluated at $x = 0$, to obtain

$$\left[\frac{\partial \hat{q}^{R'}}{\partial x} - n_0 \hat{q}^{R'} \right]_{x=0} = -2\pi m_0 \hat{B}_{T_0} \left\{ \left(1 + \frac{n_0}{2m_0}\right) T'_w - 2T'_{x=0} \right\}. \quad (30)$$

The different approaches thus lead to different boundary conditions. Equations (28) and (30) become identical, however, when $m_0/n_0 = \frac{1}{2}$. The conditions under which this equality holds will be discussed in the next section.

The difference between the two radiative boundary conditions is due to the way in which the substitute-kernel approximation has been applied in the two cases. To obtain equation (28), the approximation was applied formally throughout, i.e. the approximate equations (24) and (25) were used in a purely formal manner. Equation (30), on the other hand, was obtained from the exact relation (29) plus the approximate equations (23) and (25). This points out the type of internal inconsistency that can arise in the radiative equations when the substitute-kernel approximation is applied in different ways. We consider the formal application of the approximation to be the more self-consistent. We therefore adopt equation (28) as the radiative boundary condition.

The grey-gas counterparts of equations (27) and (28) are

$$\frac{\partial^2 q^{R'}}{\partial x^2} - 4\pi a \alpha_0 B_{T_0} \frac{\partial T'}{\partial x} - b^2 \alpha_0^2 q^{R'} = 0$$

and

$$\left[\frac{\partial q^{R'}}{\partial x} - b\alpha_0 q^{R'}\right]_{x=0} = -4\pi a \alpha_0 B_{T_0}(T'_w - T'_{x=0}),$$

where $q^{R'}$ is the heat flux over all frequency. These equations are found by specializing equations (27) and (28) to the situation where $\alpha_{\nu_0} = \alpha_0 =$ constant for all ν. With the appropriate exponential approximation for both F_2 and E_2, we then have $m_0 \exp(-n_0 z) \cong F_2 = \alpha_0 E_2(\alpha_0 z) \cong \alpha_0 a \exp(-b\alpha_0 z)$, so that $m_0 = a\alpha_0$ and $n_0 = b\alpha_0$ in the above equations.

We see that the grey and nongrey equations have the same mathematical structure. They differ only in the coefficients that appear in the various terms. Since these coefficients are constants for any given reference state, solutions previously obtained with the grey exponential approximation can therefore be reinterpreted for a nongrey gas through an appropriate normalization (see Section 5).

4. ANALYTICAL RELATIONS FOR m_0 AND n_0

The functions m_0 and n_0 can be obtained analytically in much the same way as is done for the constants a and b in the grey exponential approximation, that is, by analytically matching certain properties of the exact and approximate transmission functions.

An inspection of the equations leading to the boundary condition (28) suggests the importance of an accurate representation of F_2 and F_3 at $z = 0$. An obvious choice of matching therefore is to take $m_0 = F_2(0)$ and $m_0/n_0 = F_3(0)$ (see equations (20) and (21)). Since $F_3(0) = \int_0^\infty F_2(z)\,dz$ from the recurrence formula (17), this is equivalent to matching the zero-intercept and area of the function F_2. In making this choice we in effect emphasize the importance of the boundary condition and of the radiative field optically close to the wall. With this matching, the definition (15) of F_3 leads to

$$\frac{m_0}{n_0} = \frac{1}{2}, \tag{31a}$$

and this result and the definition (14) of F_2 give

$$n_0 = 2\frac{\displaystyle\int_{\alpha_{\nu_0}\neq 0} \alpha_{\nu_0} \left.\frac{dB_\nu}{dT}\right|_0 d\nu}{\displaystyle\int_{\alpha_{\nu_0}\neq 0} \left.\frac{dB_\nu}{dT}\right|_0 d\nu}$$

$$\equiv 2\hat{\alpha}_{LP_0} \xrightarrow[\text{for all } \nu]{\alpha_{\nu_0}\text{ nonzero}} 2\alpha_{LP_0}. \tag{31b}$$

In these we have used the fact that $E_3(0) = \frac{1}{2}$ and $E_2(0) = 1$.

The quantity $\hat{\alpha}_{LP_0}$, which is a linear Planck mean over restricted ranges of frequency, is one possible choice of a physically meaningful emission coefficient for a nongrey gas. In the situation where α_{ν_0} is nonzero for all ν, it goes over into α_{LP_0}, the linear Planck mean over all frequency, which has been discussed in detail by Cogley, Vincenti, and Gilles [7]. As in that reference, $\hat{\alpha}_{LP_0}$ can be shown to be, for the present spectral model, the correct mean emission coefficient in the exact asymptotic limit of an emission-controlled situation (optically thin gas near equilibrium with negligible radiation from the boundaries). It is also the correct mean in the less restrictive thin-gas limit near equilibrium when the boundaries are isothermal and radiate isotropically, which is the situation we treat here. (This special case was implied in [7] immediately following equation (16).) It is not surprising that the matching of the preceding paragraph introduces $\hat{\alpha}_{LP_0}$ as the mean coefficient, since we have weighted the approximation in favor of small values of $n_0 z$ and the gas will always appear thin for sufficiently small values of this product.

If we wish to emphasize the radiative field at larger distances, we can match the exponential function to the area and first moment of F_2. In view of the equivalence noted previously, the area matching leads to the same result as in equation (31a). The matching of the first moment gives, after introduction of $E_2(\alpha_{\nu_0} z)$ from equation (8) and interchange of the order of integration†,

$$\int_0^\infty m_0 z\, e^{-n_0 z}\, dz = \frac{1}{\hat{B}_{T_0}} \int_{\alpha_{\nu_0} \neq 0} \alpha_{\nu_0} \left.\frac{dB_\nu}{dT}\right|_0 \times \left\{ \int_0^1 \left[\int_0^\infty z \exp(-\alpha_{\nu_0} z/l)\, dz \right] dl \right\} d\nu.$$

Carrying out the integration then leads to

$$\frac{m_0}{n_0^2} = \frac{1}{3\hat{B}_{T_0}} \int_{\alpha_{\nu_0} \neq 0} \frac{1}{\alpha_{\nu_0}} \left.\frac{dB_\nu}{dT}\right|_0 d\nu. \qquad (32)$$

This and equation (31a) give finally for this matching

$$\frac{m_0}{n_0} = \frac{1}{2}, \qquad (33a)$$

and

$$n_0 = \frac{3}{2} \frac{\int_{\alpha_{\nu_0} \neq 0} \left.\frac{dB_\nu}{dT}\right|_0 d\nu}{\int_{\alpha_{\nu_0} \neq 0} \frac{1}{\alpha_{\nu_0}} \left.\frac{dB_\nu}{dT}\right|_0 d\nu} \equiv \tfrac{3}{2}\hat{\alpha}_{R_0} \xrightarrow[\text{for all } \nu]{\alpha_{\nu_0} \text{ nonzero}} \tfrac{3}{2}\alpha_{R_0}. \qquad (33b)$$

The quantity $\hat{\alpha}_{R_0}$, the Rosseland mean over restricted ranges of frequency, goes over into the ordinary Rosseland mean α_{R_0} in the situation where α_{ν_0} is nonzero for all ν. Again in [7], the Rosseland mean evaluated at the reference condition is shown to be the correct mean to use for a thick gas near equilibrium.

† The interchange in the order of integration here is valid only when the integration over ν has been limited to the frequency regions for which $\alpha_{\nu_0} \neq 0$. The correct evaluation of the right-hand side of this equation could be obtained for an integration over all ν, but care is then needed in evaluating the divergent inner integral when α_{ν_0} is zero. This is one of the reasons for introducing the integration convention (10).

That this matching introduces a Rosseland mean absorption coefficient might be expected, since it emphasizes the large values of $n_0 z$ and the gas will always appear optically thick for sufficiently large values of this quantity. This matching may not be appropriate when a radiating boundary at $x = 0$ plays a major role in the problem, since it will predict the heat addition to the gas incorrectly at small values of $n_0 x$. This conjecture will have to be confirmed, of course, by examining specific problems.

A third possible procedure is to match the exponential function to the zero-intercept and first moment of F_2. This may be thought of as a compromise between the two previous procedures. The two parts of this matching have been carried through in the course of the above (i.e. $m_0 = F_2(0) = \hat{\alpha}_{LP_0}$ and equation (32)). The combined results give finally

$$\frac{m_0}{n_0} = \left(\frac{1}{3}\frac{\hat{\alpha}_{LP_0}}{\hat{\alpha}_{R_0}}\right)^{\frac{1}{2}} \xrightarrow[\text{for all } \nu]{\alpha_{\nu_0} \text{ nonzero}} \left(\frac{1}{3}\frac{\alpha_{LP_0}}{\alpha_{R_0}}\right)^{\frac{1}{2}}, \qquad (34a)$$

and

$$n_0 = (3\hat{\alpha}_{LP_0}\hat{\alpha}_{R_0})^{\frac{1}{2}} \xrightarrow[\text{for all } \nu]{\alpha_{\nu_0} \text{ nonzero}} (3\alpha_{LP_0}\alpha_{R_0})^{\frac{1}{2}}. \qquad (34b)$$

For the special situation in which α_{ν_0} is nonzero for all ν, the results (34a, b) have also been obtained in [7] by arbitrarily requiring that the linearized grey differential approximation for the net radiative heat flux in three dimensions take on the correct thick- and thin-gas limits. Traugott [8] had previously introduced this kind of argument to incorporate spectral properties into the corresponding nonlinear equations.† In [7] the following differential equation for the net radiative flux (written here in one dimension) was obtained by this procedure:

$$\frac{\partial^2 q^{R'}}{\partial x^2} - 16\sigma T_0^3 \alpha_{LP_0} \frac{\partial T'}{\partial x} - 3\alpha_{LP_0}\alpha_{R_0} q^{R'} = 0.$$

Comparison of this equation with equation

† Both papers can be generalized by introducing means over restricted frequency rangers, as in the present formulation.

(27) in the situation where α_{ν_0} is nonzero for all ν results in the values for m_0 and n_0 given by the relations (34a, b).

The above matching procedures are the three most obvious of the many that could be devised. It is not possible without further study to say which of the three gives the best matching or to recommend the use of one particular procedure. By the assumption of suitable functional forms for α_{ν_0}, closed-form integration of the exact transmission functions may be possible. If so, an analytical comparison of the transmission functions and their exponential approximations could be made. If not, the approach must be, as stated earlier, to compare the exponential approximation with the results from numerical evaluation of the transmission functions for specific gases and specific values of T_0 and ρ_0. The first two matchings are appealing because the matching of the zero-intercept of F_3 (or equivalently the area of F_2) leads to the result $m_0/n_0 = \frac{1}{2}$. We then have no inconsistency in the radiative boundary conditions (28) and (30). The third matching does not have this property. It does, on the other hand, make the resulting differential equation for $\hat{q}^{R'}$ satisfy the correct thick- and thin-gas limits.

5. EQUATIONS OF RADIATIVE ACOUSTICS

We can now couple the equations for $\hat{q}^{R'}$ with the linearized equations of gas dynamics. For an imperfect gas in local thermodynamic equilibrium, the one-dimensional unsteady-flow equations, linearized about a uniform, equilibrium state of rest, can be written as

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} = 0,$$

$$\rho_0 \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0,$$

$$\rho_0 \frac{\partial h'}{\partial t} - \frac{\partial p'}{\partial t} = -\frac{\partial \hat{q}^{R'}}{\partial x},$$

$$h' = h_{p_0} p' + h_{\rho_0} \rho',$$

and

$$T' = T_{p_0} p' + T_{\rho_0} \rho',$$

where t is the time and u', h', p', and ρ' are the perturbations in the velocity, specific enthalpy, pressure, and density, respectively. Radiative pressure and energy density have been neglected in writing these equations. The final two equations contain the linear terms from Taylor's-series expansions of the general equilibrium state functions $h = h\,(p, \rho)$ and $T = T(p, \rho)$; the subscript notation denotes partial differentiation of these functions, for example, $h_{\rho_0} \equiv (\partial h/\partial \rho)_p|_0$.

To aid in the interpretation of the later equations, we introduce the following normalization:

$$\bar{\rho} \equiv \rho'/\rho_0, \quad \bar{T} \equiv T'/T_0, \quad \bar{p} \equiv p'/p_0, \quad \bar{u} = u'/a_{S_0},$$
$$\bar{h} \equiv h'/c_{p_0}T_0, \quad \bar{q}^R \equiv \hat{q}^{R'}/\rho_0 a_{S_0} c_{p_0} T_0, \tag{35}$$
$$\xi = n_0 x, \quad \text{and} \quad \tau = n_0 a_{S_0} t.$$

Here a_{S_0} is the isentropic speed of sound defined by

$$a_{S_0}^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_S \bigg|_0 = -\frac{h_{\rho_0}}{h_{p_0} - 1/\rho_0},$$

and c_{p_0} is the specific heat at constant pressure as given by

$$c_{p_0} = \left(\frac{\partial h}{\partial T}\right)_p \bigg|_0 = \frac{h_{\rho_0}}{T_{\rho_0}}.$$

These quantities are defined here for a gas in local thermodynamic equilibrium and are not to be confused with the corresponding quantities as defined for a chemically frozen gas (cf. [3], p. 257 for definition of the frozen and equilibrium speeds of sound).

With the normalization (35), the gas-dynamic equations become

$$\frac{\partial \bar{\rho}}{\partial \tau} + \frac{\partial \bar{u}}{\partial \xi} = 0, \tag{36}$$

$$a_{S_0}^2 \frac{\partial \bar{u}}{\partial \tau} + \frac{p_0}{\rho_0}\frac{\partial \bar{p}}{\partial \xi} = 0, \tag{37}$$

$$c_{p_0} T_0 \frac{\partial \bar{h}}{\partial \tau} - \frac{p_0}{\rho_0} \frac{\partial \bar{p}}{\partial \tau} = -c_{p_0} T_0 \frac{\partial \bar{q}^R}{\partial \xi}, \tag{38}$$

$$c_{p_0} T_0 \bar{h} = p_0 h_{p_0} \bar{p} + \rho_0 h_{\rho_0} \bar{\rho} \tag{39}$$

and

$$T_0 \bar{T} = p_0 T_{p_0} \bar{p} + \rho_0 T_{\rho_0} \bar{\rho}. \tag{40}$$

The differential equation (27) for the net radiative heat flux and the radiative boundary condition (28) become

$$\frac{\partial^2 \bar{q}^R}{\partial \xi^2} - \frac{16}{(n_0/m_0)\hat{B}o} \frac{\partial \bar{T}}{\partial \xi} - \bar{q}^R = 0 \tag{41}$$

and

$$\left[\frac{\partial \bar{q}^R}{\partial \xi} - \bar{q}^R\right]_{\xi=0} = -\frac{16}{(n_0/m_0)\hat{B}o}(\bar{T}_w - \bar{T}_{\xi=0}), \tag{42}$$

where the nongrey Boltzmann number is defined by

$$\hat{B}o \equiv \frac{\rho_0 a_{S_0} c_{p_0}}{(\pi/4)\hat{B}_{T_0}} \xrightarrow[\text{for all } \nu]{\alpha_{\nu_0}\text{ nonzero}} \frac{\rho_0 a_{S_0} c_{p_0}}{(\pi/4) B_{T_0}} = \frac{\rho_0 a_{S_0} c_{p_0}}{\sigma T_0^3} \equiv Bo. \tag{43}$$

If we introduce the normalized perturbation potential $\bar{\varphi}$ defined by $\bar{u} = \partial\bar{\varphi}/\partial\xi$ and $(p_0/\rho_0 a_{S_0}^2)\,\bar{p} = -\partial\bar{\varphi}/\partial\tau$ (thereby satisfying equation (37)) and eliminate $\bar{\rho}$, $\bar{h}$, $\bar{T}$, and $\bar{q}^R$ between the remaining equations (36) through (41) (cf. [3], p. 497), we obtain the following fifth-order partial differential equation:

$$(\bar{\varphi}_{\tau\tau} - \bar{\varphi}_{\xi\xi})_{\xi\xi\tau} + \frac{16}{(n_0/m_0)\hat{B}_0}\left(\frac{a_{S_0}^2}{a_{T_0}^2}\bar{\varphi}_{\tau\tau} - \bar{\varphi}_{\xi\xi}\right)_{\xi\xi} - (\bar{\varphi}_{\tau\tau} - \bar{\varphi}_{\xi\xi})_\tau = 0. \tag{44}$$

The subscripts ξ and τ denote partial derivatives, and the isothermal speed of sound a_{T_0} is defined by

$$a_{T_0}^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_T\bigg|_0 = -T_{\rho_0}/T_{p_0}.$$

From equation (42), we can in a similar manner write the radiative boundary condition on $\bar{T}_w(\tau) = T'_w(\tau)/T_0$ as

$$\frac{d\bar{T}_w(\tau)}{d\tau} = \frac{(n_0/m_0)\hat{B}o}{16}[(W_S)_\xi - W_S]_\tau|_{\xi=0} + [(W_T)_\xi - W_T]|_{\xi=0}, \tag{45}$$

where we have for brevity introduced the notation

$$W_S \equiv \bar{\varphi}_{\tau\tau} - \bar{\varphi}_{\xi\xi}$$

and

$$W_T \equiv \frac{a_{S_0}^2}{a_{T_0}^2}\bar{\varphi}_{\tau\tau} - \bar{\varphi}_{\xi\xi}.$$

Values for m_0 and n_0 need not be specified in order to carry out generalized solutions based on equations (44) and (45). This is so because the dimensionless parameters $\hat{B}o$ and n_0/m_0 always appear as a product and can therefore be included into a single new parameter $K \equiv (n_0/m_0)\hat{B}o$. The ratio n_0/m_0 appears with the Boltzmann number as a consequence of the nongrey substitute-kernel approximation.

The nongrey Boltzmann number $\hat{B}o$ retains the same physical meaning that the ordinary Boltzmann number Bo (see equation (43)) carries throughout the literature. Its interpretation, however, is slightly different owing to the fact that the present radiative model may represent a gas that emits and absorbs radiant energy in finite intervals of frequency. If we multiply the numerator and denominator of definition (43) by T' we obtain

$$\hat{B}o = \frac{\rho_0 a_{S_0} c_{p_0} T'}{(\pi/4)\hat{B}_{T_0} T'},$$

which can be interpreted as a measure of the ratio of the energy flux of the wave to the radiant energy flux due to spontaneous emission from the general nongrey gas. The definition of Bo, when written in the above manner, contains in its denominator the term $(\pi/4)\,B_{T_0}T' = \sigma T_0^3 T'$, which characterizes the spontaneous emission only in the special situation where α_{ν_0} is nonzero for all ν (see equation (43)).

As with the radiative equations in section 3, the acoustic equation (44) and radiative boundary condition (45) have the same structure as the corresponding equations for a grey gas (cf. [3], p. 491 and 499). They differ only in the definitions of the normalized independent variables and the Boltzmann number. Solutions already carried out with the exponential approximation for a grey gas can be reinterpreted accordingly for a nongrey gas by replacing the grey-gas parameters $a\alpha_0$, $b\alpha_0$ and Bo by their nongrey counterparts m_0, n_0, and $\hat{B}o$, respectively.

6. CONCLUDING REMARKS

The above development depends critically on the linearization of the equation of radiative transfer. There is a possibility, however, that the basic ideas may carry over to the nonlinear problem. Such generalization would entail certain restrictions on α_ν and B_ν, so that the integrations over frequency and position could again be carried out independently. These restrictions might do less violence to reality than the drastic assumption of a grey gas, thus leading to a more accurate description of radiative transfer for use in analytical studies.

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Résumé—Les équations pour le transport par rayonnement dans des géométries à plaques parallèles sont etudiées pour un gaz non gris près de l'équilibre. On suppose que les processus moléculaires s'effectuent avec un équilibre thermodynamique local. En supposant que les déviations à partir d'un état de référence d'équilibre de rayonnement sont faibles, on linéarise l'équation du transport par rayonnement. Ceci permet d'effectuer indépendamment les intégrations nécessaires par rapport à l'espace et à la fréquence spatiale. Par analogie avec les procédés du gaz gris, une approximation de noyau de remplacement non gris (ou exponentiel) est alors faite pour certaines fonctions de transmission intégrées par rapport à la fréquence qui se trouvent dans les expressions des flux de chaleur. Les propriétés spectrales apparaissent dans la formulation dans deux fonctions, qui sont introduites par l'approximation et qui dépendent de l'état de référence du gaz. Ces fonctions sont obtenues par des procédés de jonction analytique, qui définissent les coefficients d'absorption moyens linéarisés de Planck et de Rosseland ayant un sens physique pour un gaz général non gris. Pour l'emploi en rayonnement acoustique, l'équation différentielle pour le flux de chaleur est couplée avec les équations linéarisées de la dynamique des gaz. Les équations résultantes pour un gaz non gris ont la même structure mathématique que les équations pour un gaz gris, qui sont maintenant comprises comme cas spécial. Les résultats des solutions existantes du gaz gris peuvent donc être réinterprétées à l'aide d'un gaz non gris par une normalisation appropriée.

Zusammenfassung—Die Gleichungen für den Wärmeaustausch durch Strahlung in geometrisch planparallelen Anordnungen sind untersucht für ein nicht graues Gas in der Nähe des Gleichgewichts. In den molekularen Prozessen wurde lokales thermodynamisches Gleichgewicht vorausgesetzt. Unter der Annahme, dass nur geringe Abweichungen von einem Bezugszustand des Strahlungsgleichgewichts vorkommen, wurde die Gleichung für den Wärmeaustausch durch Strahlung linearisiert. Dies erlaubt eine unabhängige Durchführung der erforderlichen Integrationen über den Raum und über die Spektralfrequenz. In Analogie zu den Rechenverfahren beim grauen Gas wurde dann beim nicht-grauen Gas eine Ersatzkernfunktion- (oder exponentielle) Näherung gemacht für bestimmte über die Frequenz integrierte Übertragungsfunktionen, die bei den Ausdrücken für die Wärmeströme auftauchen.

Die spektalen Eigenschaften erscheinen bei diesem Ansatz in zwei Funktionen, die durch die Näherung eingeführt wurden, und die vom Bezugszustand des Gases abhängen. Diese Funktionen wurden durch analytischen Vergleich von Prozeduren ermittelt, die linearisierte mittlere Absorptionskoeffizienten (ähnlich den Planck- und Rolleland'schen Koeffizienten) definieren, die für ein allgemeines nicht-graues Gas physikalisch bedeutsam sind. Zur Anwendung in der Strahlungsakustik wurde die Differentialgleichung für den Wärmestrom gekoppelt mit den linearisierten Gleichungen der Gasdynamik. Die für das nicht-graue Gas resultierenden Gleichungen haben die selbe mathematische Struktur, wie die Gleichungen für das graue Gas, die nun als Spezialfall enthalten sind. Die Ergebnisse der existierenden Lösungen für das graue Gas können deshalb durch zweckmässige Vorschriften für das nicht-graue Gas uminterpretiert werden.

Аннотация—Изучаются уравнения лучистого переноса в плоскопараллельных геометриях для несерого газа в состоянии; близком к равновесному. Предполагается, что в молекулярных процессах имеет место локальное термодинамическое равновесие. При допущении, что отклонения от начального состояния лучистого равновесия невелики, линеаризировано уравнение лучистого переноса. Это позволило провести независимое интегрирование по пространству и спектральной частоте. По аналогии с методикой, используемой для серого газа, для несерого газа при использовании аппроксимации с заменой ядра (или экспоненциальной) затем находится преобразование для определенных частотно-интегрируемых функций, которые встречаются в выражениях для тепловых потоков. Спектральные свойства представлены в формуле двумя функциями, которые являются результатом принятой аппроксимации и которые зависят от начального состояния газа. Эти функции находятся методами аналитической подгонки, которые определяют линеаризированные средние коэффициенты абсорбции Планка, Россланда, имеющие физический смысл для обычного несерого газа. Для использования в лучистой акустике, дифференциальное уравнение теплового потока объединяется с линеаризированными уравнениями газодинамики. Получаемые в результате уравнения для несерого газа имеют ту же математическую структуру, что и удавнения для серого газа, которые теперь являются частным случаем. Результаты имеющихся решений для серого газа можно поэтому выразить с помощью выражений для несерого газа путем соответствующей нормализации.